

# A note of pointwise estimates on Shishkin meshes

Jin Zhang<sup>\*†</sup>

January 30, 2012

## Abstract

We propose the estimates of the discrete Green function for the stream-line diffusion finite element method (SDFEM) on Shishkin meshes.

## 1 Problem

We consider the singularly perturbed boundary value problem

$$(1.1a) \quad Lu := -\varepsilon \Delta u + \mathbf{b} \cdot \nabla u + u = f \quad \text{in} \quad \Omega = (0, 1)^2,$$

$$(1.1b) \quad u = 0 \quad \text{on} \quad \partial\Omega,$$

where  $\varepsilon \ll 1$  is a small positive parameter and  $\mathbf{b} = (b_1, b_2)^T > (0, 0)^T$  is constant. It is also assumed that  $f$  is sufficiently smooth.

## 2 The SDFEM on Shishkin meshes

### 2.1 Shishkin meshes

Let  $N > 4$  be a positive even integer. We use a piecewise uniform mesh — a so-called *Shishkin* mesh — with  $N$  mesh intervals in both  $x$ - and  $y$ -direction which condenses in the layer regions. For this purpose we define the two mesh transition parameters

$$\lambda_x := \min \left\{ \frac{1}{2}, 2 \frac{\varepsilon}{\beta_1} \ln N \right\} \quad \text{and} \quad \lambda_y := \min \left\{ \frac{1}{2}, 2 \frac{\varepsilon}{\beta_2} \ln N \right\}.$$

**Assumption 1.** *We assume in our analysis that  $\varepsilon \leq N^{-1}$ , as is generally the case in practice. Furthermore we assume that  $\lambda_x = 2\varepsilon\beta_1^{-1} \ln N$  and  $\lambda_y = 2\varepsilon\beta_2^{-1} \ln N$  as otherwise  $N^{-1}$  is exponentially small compared with  $\varepsilon$ .*

---

<sup>\*</sup>Email: JinZhangalex@hotmail.com

<sup>†</sup>Address: School of Science, Xi'an Jiaotong University, Xi'an, 710049, China

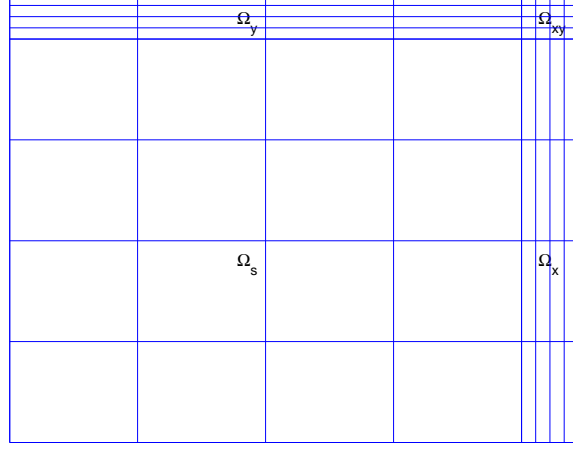


Figure 1: Dissection of  $\Omega$  and triangulation  $\Omega^N$ .

The domain  $\Omega$  is dissected into four parts as  $\Omega = \Omega_s \cup \Omega_x \cup \Omega_y \cup \Omega_{xy}$  (see FIG. 1), where

$$\begin{aligned}\Omega_s &:= [0, 1 - \lambda_x] \times [0, 1 - \lambda_y], & \Omega_x &:= [1 - \lambda_x, 1] \times [0, 1 - \lambda_y], \\ \Omega_y &:= [0, 1 - \lambda_x] \times [1 - \lambda_y, 1], & \Omega_{xy} &:= [1 - \lambda_x, 1] \times [1 - \lambda_y, 1].\end{aligned}$$

We introduce the set of mesh points  $\{(x_i, y_j) \in \Omega : i, j = 0, \dots, N\}$  defined by

$$x_i = \begin{cases} 2i(1 - \lambda_x)/N, & \text{for } i = 0, \dots, N/2, \\ 1 - 2(N - i)\lambda_x/N, & \text{for } i = N/2 + 1, \dots, N \end{cases}$$

and

$$y_j = \begin{cases} 2j(1 - \lambda_y)/N, & \text{for } j = 0, \dots, N/2, \\ 1 - 2(N - j)\lambda_y/N, & \text{for } j = N/2 + 1, \dots, N. \end{cases}$$

By drawing lines through these mesh points parallel to the  $x$ -axis and  $y$ -axis the domain  $\Omega$  is partitioned into rectangles. This triangulation is denoted by  $\Omega^N$  (see FIG. 1). If  $D$  is a mesh subdomain of  $\Omega$ , we write  $D^N$  for the triangulation of  $D$ . The mesh sizes  $h_{x,\tau} = x_i - x_{i-1}$  and  $h_{y,\tau} = y_j - y_{j-1}$  satisfy

$$h_{x,\tau} = \begin{cases} H_x := \frac{1 - \lambda_x}{N/2}, & \text{for } i = 1, \dots, N/2, \\ h_x := \frac{\lambda_x}{N/2}, & \text{for } i = N/2 + 1, \dots, N \end{cases}$$

and

$$h_{y,\tau} = \begin{cases} H_y := \frac{1-\lambda_y}{N/2}, & \text{for } j = 1, \dots, N/2, \\ h_y := \frac{\lambda_y}{N/2}, & \text{for } j = N/2 + 1, \dots, N. \end{cases}$$

The mesh sizes  $h_{x,\tau}$  and  $h_{y,\tau}$  satisfy

$$N^{-1} \leq H_x, H_y \leq 2N^{-1} \quad \text{and} \quad C_1 \varepsilon N^{-1} \ln N \leq h_x, h_y \leq C_2 \varepsilon N^{-1} \ln N,$$

where  $C_1$  and  $C_2$  are positive constants and independent of  $\varepsilon$  and of the mesh parameter  $N$ . The above properties are essential when inverse inequalities are applied in our later analysis.

For the mesh elements we shall use two notations:  $\tau_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$  for a specific element, and  $\tau$  for a generic mesh rectangle.

## 2.2 The streamline diffusion finite element method

Let  $V := H_0^1(\Omega)$ . On the above Shishkin mesh we define a finite element space

$$V^N := \{v^N \in C(\bar{\Omega}) : v^N|_{\partial\Omega} = 0 \text{ and } v^N|_{\tau} \text{ is bilinear, } \forall \tau \in \Omega^N\}.$$

In this case, the SDFEM reads as

$$(2.1) \quad \begin{cases} \text{Find } U \in V^N \text{ such that for all } v^N \in V^N \\ \varepsilon(\nabla U, \nabla v^N) + (\mathbf{b} \cdot \nabla U + U, v^N + \delta \mathbf{b} \cdot \nabla v^N) = (f, v^N + \delta \mathbf{b} \cdot \nabla v^N). \end{cases}$$

where  $\delta = \delta(\mathbf{x})$  is a user-chosen parameter (see [3]).

We set

$$b := \sqrt{b_1^2 + b_2^2}, \quad \boldsymbol{\beta} := \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} / b, \quad \boldsymbol{\eta} := \begin{pmatrix} -b_2 \\ b_1 \end{pmatrix} / b \quad \text{and} \quad v_\zeta := \boldsymbol{\zeta}^T \nabla v$$

for any vector  $\boldsymbol{\zeta}$  of unit length. By an easy calculation one shows that

$$(\nabla w, \nabla v) = (w_\beta, v_\beta) + (w_\eta, v_\eta).$$

We rewrite (2.1) as

$$\varepsilon(U_\beta, v_\beta^N) + \varepsilon(U_\eta, v_\eta^N) + (bU_\beta + U, v^N + \delta b v_\beta^N) = (f, v^N + \delta b v_\beta^N)$$

and, following usual practice, we set

$$\delta(\mathbf{x}) := \begin{cases} N^{-1}, & \text{if } \mathbf{x} \in \Omega_s, \\ 0, & \text{otherwise.} \end{cases}$$

For technical reasons in the later analysis, we increase the crosswind diffusion(see [4]) by replacing  $\varepsilon(U_\eta, v_\eta^N)$  by  $\hat{\varepsilon}(U_\eta, v_\eta^N)$  where

$$\hat{\varepsilon} := \max(\varepsilon, N^{-3/2})$$

and

$$\hat{\varepsilon}(\mathbf{x}) := \begin{cases} \tilde{\varepsilon}, & \mathbf{x} \in \Omega_s, \\ \varepsilon, & \mathbf{x} \in \Omega \setminus \Omega_s. \end{cases}$$

We now state our streamline diffusion method with artificial crosswind:

$$(2.2) \quad \begin{cases} \text{Find } U \in V^N \text{ such that for all } v^N \in V^N \\ B(U, v^N) = (f, v^N + \delta b v_\beta^N), \end{cases}$$

with

$$(2.3) \quad B(U, v^N) := (\varepsilon + b^2 \delta)(U_\beta, v_\beta^N) + \hat{\varepsilon}(U_\eta, v_\eta^N) - b(1 - \delta)(U, v_\beta^N) + (U, v^N).$$

### 3 The discrete Green function

Let  $\mathbf{x}^*$  be a mesh node in  $\Omega$ . The discrete Green's function  $G \in V^N$  associated with  $\mathbf{x}^*$  is defined by

$$B(v^N, G) = v^N(\mathbf{x}^*), \forall v^N \in V^N.$$

The weighted function  $\omega$ :

$$\omega(\mathbf{x}) := g\left(\frac{(\mathbf{x} - \mathbf{x}^*) \cdot \boldsymbol{\beta}}{\sigma_\beta}\right) g\left(\frac{(\mathbf{x} - \mathbf{x}^*) \cdot \boldsymbol{\eta}}{\sigma_\eta}\right) g\left(-\frac{(\mathbf{x} - \mathbf{x}^*) \cdot \boldsymbol{\eta}}{\sigma_\eta}\right)$$

where

$$g(r) = \frac{2}{1 + e^r} \quad \text{for } r \in (-\infty, \infty).$$

and  $\sigma_\beta = kN^{-1} \ln N$  and  $\sigma_\eta = k\tilde{\varepsilon}^{1/2} \ln N$ .

$$|||G|||_\omega^2 := (\varepsilon + b^2 \delta) \|\omega^{-1/2} G_\beta\|^2 + \hat{\varepsilon} \|\omega^{-1/2} G_\eta\|^2 + \frac{b}{2} \|(\omega^{-1})_\beta^{1/2} G\|^2 + \|\omega^{-1/2} G\|^2$$

and

$$(3.1) \quad \begin{aligned} |||G|||_\omega^2 &= B(\omega^{-1} G, G) - (\varepsilon + b^2 \delta)((\omega^{-1})_\beta G, G_\beta) \\ &\quad - \hat{\varepsilon}((\omega^{-1})_\eta G, G_\eta) - b\delta(\omega^{-1} G, G_\beta). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} B(\omega^{-1} G, G) &= B(E, G) + B((\omega^{-1} G)^I, G) \\ &= B(E, G) + (\omega^{-1} G)(\mathbf{x}^*) \end{aligned}$$

where  $E := \omega^{-1} G - (\omega^{-1} G)^I$ .

**Lemma 1.** *If  $\sigma_\beta = kN^{-1} \ln N$  and  $\sigma_\eta = k \ln N \tilde{\varepsilon}^{1/2}$ , then for  $k > 1$  sufficiently large and independent of  $N$  and  $\varepsilon$ , we have*

$$B(\omega^{-1} G, G) \geq \frac{1}{4} |||G|||_\omega^2.$$

*Proof.* From (3.1), we estimate the following terms.

$$\begin{aligned} (\varepsilon + \delta) |((\omega^{-1})_\beta G, G_\beta)| &\leq C(\varepsilon + \delta)^{1/2} \sigma_\beta^{-1/2} \cdot \|(\omega^{-1})_\beta^{1/2} G\| \cdot (\varepsilon + \delta)^{1/2} \|\omega^{-1/2} G_\beta\| \\ &\leq C(\varepsilon + \delta)^{1/2} \sigma_\beta^{-1/2} \|G\|_\omega^2 \end{aligned}$$

and

$$\begin{aligned} \hat{\varepsilon} |((\omega^{-1})_\eta G, G_\eta)| &\leq C\hat{\varepsilon}^{1/2} \sigma_\eta^{-1} \cdot \|\omega^{-1/2} G\| \cdot \hat{\varepsilon}^{1/2} \|\omega^{-1/2} G_\eta\| \\ &\leq C\hat{\varepsilon}^{1/2} \sigma_\eta^{-1} \|G\|_\omega^2 \end{aligned}$$

For  $b\delta(\omega^{-1}G, G_\beta)$ , we make use of integration by parts.

From the definition of  $\sigma_\beta$  and  $\sigma_\eta$  and  $\varepsilon \leq N^{-1}$ , we take  $k$  sufficiently large and we are done.  $\square$

**Lemma 2.** *If  $\sigma_\beta = kN^{-1} \ln N$ , with  $k > 0$  sufficiently large and independent of  $N$  and  $\varepsilon$ . Then for each mesh point  $\mathbf{x}^* \in \Omega \setminus \Omega_{xy}$ , we have*

$$|(\omega^{-1}G)(\mathbf{x}^*)| \leq \frac{1}{16} \|G\|_\omega^2 + CN \ln N.$$

where  $C$  is independent of  $N$ ,  $\varepsilon$  and  $\mathbf{x}^*$ .

*Proof.* First let  $\mathbf{x}^* \in \Omega_s$ . Let  $\tau^*$  be the unique triangle that has  $\mathbf{x}^*$  as its north-east corner. Then

$$\begin{aligned} |(\omega^{-1}G)(\mathbf{x}^*)| &\leq CN \|G\|_{\tau^*} \\ &\leq CN \max_{\tau^*} |(\omega^{-1})_\beta^{-1/2}| \cdot \|(\omega^{-1})_\beta^{-1/2} G\|_{\tau^*} \end{aligned}$$

Calculating  $(\omega^{-1})_\beta^{-1}(\mathbf{x})$  explicitly, we see that

$$(\omega^{-1})_\beta^{-1}(\mathbf{x}) \leq C\sigma_\beta = CkN^{-1} \ln N \quad \forall \mathbf{x} \in \tau^*$$

Thus

$$|(\omega^{-1}G)(\mathbf{x}^*)| \leq CN \ln N + \frac{1}{16} \|G\|_\omega^2$$

by means of the arithmetic-geometric mean inequality.

Next, let  $\mathbf{x}^* \in \Omega_x$ . (The case  $\mathbf{x}^* \in \Omega_y$  is similar.) Write  $\mathbf{x}^* = (x_i, y_j)$ . Then

$$\begin{aligned} |\omega^{-1}G(\mathbf{x}^*)| &= |G(\mathbf{x}^*)| \\ &= \left| \int_{x_i}^1 G_x(t, y_j) dt \right| \\ &\leq CH_y^{-1} \int_{x_i}^1 \int_{y_j}^{y_{j+1}} |G_x(t, y)| dy dt \\ &\leq CN(\varepsilon \ln N \cdot N^{-1})^{1/2} \|G_x\|_{\Omega_x} \\ &\leq CN^{1/2} \ln^{1/2} N \|G\| \end{aligned}$$

where  $G_x(t, y_j) = \frac{G_{k,j} - G_{k-1,j}}{h_x}$  for  $(t, y_j) \in \tau_{kj}$ .

Analysis: for the relation of boundary integral and domain integral, we analyze

$$\int_{x_i}^1 \int_{y_j}^{y_{j+1}} |G_x(t, y)| dy dt = \sum_{k=i+1}^N h_x \int_{y_j}^{y_{j+1}} \left| f(y_j) \frac{y_{j+1} - y}{H_y} + f(y_{j+1}) \frac{y - y_j}{H_y} \right| dy$$

where  $f(y_j) = \frac{G_{k,j} - G_{k-1,j}}{h_x}$  and  $f(y_{j+1}) = \frac{G_{k,j+1} - G_{k-1,j+1}}{h_x}$ .

For  $\Delta_1 := \int_{y_j}^{y_{j+1}} \left| f(y_j) \frac{y_{j+1} - y}{H_y} + f(y_{j+1}) \frac{y - y_j}{H_y} \right| dy$ , we have

$$\Delta_1 = \begin{cases} \frac{|f(y_j)| + |f(y_{j+1})|}{2} H_y \geq \frac{1}{2} \max\{|f(y_j)|, |f(y_{j+1})|\} H_y & \text{if } f(y_j)f(y_{j+1}) \geq 0 \\ \frac{1}{2} \frac{f^2(y_j) + f^2(y_{j+1})}{|f(y_{j+1}) - f(y_j)|} H_y \geq \frac{1}{4} \max\{|f(y_j)|, |f(y_{j+1})|\} H_y & \text{if } f(y_j)f(y_{j+1}) < 0 \end{cases}$$

□

For  $\forall v \in C^2(\tau)$ , we have

$$|v_x| \leq C(|v_\beta| + |v_\eta|)$$

$$|v_{xx}| \leq C(|v_{\beta\beta}| + |v_{\beta\eta}| + |v_{\eta\eta}|)$$

Similarly, we have

$$|v_\beta| \leq C(|v_x| + |v_y|)$$

$$|v_{\beta\beta}| \leq C(|v_{xx}| + |v_{xy}| + |v_{yy}|)$$

**Lemma 3.** Let  $\tau \in \Omega^N$ . Then

$$\begin{aligned} \|\omega^{1/2} D^\alpha E\|_{\Omega_s} &\leq C k^{-1/2} N^{1/2} \|G\|_\omega \\ \|\omega^{1/2} D^\alpha E\|_{\Omega^N \setminus \Omega_s} &\leq C k^{-1} \varepsilon^{-1/2} \ln^{-1} N \|G\|_\omega \end{aligned}$$

where  $H_\tau = \max\{h_{x,\tau}, h_{y,\tau}\}$  and  $|\alpha| = 1$ ,  $D^\alpha G$  are  $G_\beta$  and  $G_\eta$ .

*Proof.* Assume  $p \in [1, \infty]$  and  $g \in C^3(\tau)$ . Then (see [2, Theorem 4])

$$\begin{aligned} \|(g - g^I)_x\|_{L^p(\tau)} &\leq C(h_{x,\tau}^2 \|g_{xxx}\|_{L^p(\tau)} + h_{x,\tau} h_{y,\tau} \|g_{xxy}\|_{L^p(\tau)} + h_{y,\tau}^2 \|g_{xyy}\|_{L^p(\tau)}) \\ &\quad + C h_{x,\tau} \|g_{xx}\|_{L^p(\tau)}, \\ \|(g - g^I)_y\|_{L^p(\tau)} &\leq C(h_{x,\tau}^2 \|g_{xyy}\|_{L^p(\tau)} + h_{x,\tau} h_{y,\tau} \|g_{xyy}\|_{L^p(\tau)} + h_{y,\tau}^2 \|g_{yyy}\|_{L^p(\tau)}) \\ &\quad + C h_{y,\tau} \|g_{yy}\|_{L^p(\tau)}. \end{aligned}$$

or (see [1, Comment 2.15])

$$\begin{aligned} \|(g - g^I)_x\|_{L^2(\tau)} &\leq C h_{y,\tau}^2 \|g_{xyy}\|_{L^2(\tau)} + C h_{x,\tau} \|g_{xx}\|_{L^2(\tau)}, \\ \|(g - g^I)_y\|_{L^2(\tau)} &\leq C h_{x,\tau}^2 \|g_{xyy}\|_{L^2(\tau)} + C h_{y,\tau} \|g_{yy}\|_{L^2(\tau)}. \end{aligned}$$

In the following analysis,  $D^\alpha v$  denotes the directional derivative of  $v$  along  $\beta$  or  $\eta$  for different orders. The following analysis makes use of the former estimates (The latter will make the analysis more shorter).

For  $\tau \in \Omega^N$ , we have

$$\begin{aligned}
\|\omega^{1/2}E_\beta\|_\tau &\leq C\max_\tau \omega^{1/2}(\|E_x\|_\tau + \|E_y\|_\tau) \\
&\leq C\max_\tau \omega^{1/2}\{h_{x,\tau}^2\|(\omega^{-1}G)_{xxx}\|_\tau + h_{x,\tau}\|(\omega^{-1}G)_{xx}\|_\tau + h_{x,\tau}H_\tau\|(\omega^{-1}G)_{xy}\|_\tau \\
&\quad + H_\tau h_{y,\tau}\|(\omega^{-1}G)_{xyy}\|_\tau + h_{y,\tau}^2\|(\omega^{-1}G)_{yyy}\|_\tau + h_{y,\tau}\|(\omega^{-1}G)_{yy}\|_\tau\} \\
&\leq Ch_{x,\tau}^2(\|(\omega^{-1})_{xxx}G\|_\tau + \|(\omega^{-1})_{xx}G_x\|_\tau) + Ch_{x,\tau}(\|(\omega^{-1})_{xx}G\|_\tau + \|(\omega^{-1})_xG_x\|_\tau) \\
&\quad + Ch_{y,\tau}^2(\|(\omega^{-1})_{yyy}G\|_\tau + \|(\omega^{-1})_{yy}G_y\|_\tau) + Ch_{y,\tau}(\|(\omega^{-1})_{yy}G\|_\tau + \|(\omega^{-1})_yG_y\|_\tau) \\
&\quad + Ch_{x,\tau}H_\tau(\|(\omega^{-1})_{xyy}G\|_\tau + \|(\omega^{-1})_{xy}G_y\|_\tau + \|(\omega^{-1})_{xy}G_x\|_\tau + \|(\omega^{-1})_xG_{xy}\|_\tau) \\
&\quad + H_\tau h_{y,\tau}(\|(\omega^{-1})_{xyy}G\|_\tau + \|(\omega^{-1})_{xy}G_y\|_\tau + \|(\omega^{-1})_{yy}G_x\|_\tau + \|(\omega^{-1})_yG_{xy}\|_\tau) \\
&\leq CH_\tau^2 \sum_{k=2}^3 \sum_{\substack{|\alpha|+|\gamma|=3 \\ |\alpha|\geq k}} \|D^\alpha(\omega^{-1})D^\gamma G\|_\tau + CH_\tau \sum_{k=1}^2 \sum_{\substack{|\alpha|+|\gamma|=2 \\ |\alpha|\geq k}} \|D^\alpha(\omega^{-1})D^\gamma G\|_\tau
\end{aligned}$$

where we have used the following analysis for  $\sum_{|\alpha|=1, |\gamma|=2} D^\alpha(\omega^{-1})D^\gamma G$  :

$$\begin{aligned}
\|(\omega^{-1})_xG_{xy}\|_\tau &\leq C\| \sum_{|\alpha|=1} |D^\alpha(\omega^{-1})| \cdot G_{xy} \|_\tau \\
&\leq C\max_\tau \sum_{|\alpha|=1} |D^\alpha(\omega^{-1})| \cdot \|G_{xy}\|_\tau \\
&\leq Ch_{y,\tau}^{-1}\max_\tau \sum_{|\alpha|=1} |D^\alpha(\omega^{-1})| \cdot \|G_x\|_\tau \\
&\leq Ch_{y,\tau}^{-1}\| \sum_{|\alpha|=1} |D^\alpha(\omega^{-1})| \cdot G_x \|_\tau.
\end{aligned}$$

The same analysis can be applied to  $\|\omega^{1/2}E_\eta\|_\tau$ .

For  $\tau \in \Omega_s$ , we have

$$\begin{aligned}
\|\omega^{1/2}E_\beta\|_\tau &\leq Ck^{-5/2}N^{-2}[(\sigma_\beta^{-5/2} + \sigma_\beta^{-3/2}\sigma_\eta^{-1} + \sigma_\beta^{-1/2}\sigma_\eta^{-2})\max_\tau(\omega^{-1})_\beta^{1/2} + \sigma_\eta^{-3}\max_\tau\omega^{-1/2}]\|G\|_\tau \\
&\quad + Ck^{-3/2}H_\tau^2[(\sigma_\beta^{-3/2} + \sigma_\beta^{-1/2}\sigma_\eta^{-1})\max_\tau(\omega^{-1})_\beta^{1/2} + \sigma_\eta^{-2}\max_\tau\omega^{-1/2}] \cdot N\|G\|_\tau \\
&\quad + Ck^{-3/2}H_\tau[(\sigma_\beta^{-3/2} + \sigma_\beta^{-1/2}\sigma_\eta^{-1})\max_\tau(\omega^{-1})_\beta^{1/2} + \sigma_\eta^{-2}\max_\tau\omega^{-1/2}]\|G\|_\tau \\
&\quad + C\max_\tau \omega^{1/2}H_\tau \sum_{|\alpha|=1} \|D^\alpha(\omega^{-1})\|_{L^\infty(\tau)} \cdot \sum_{|\gamma|=1} \|D^\gamma G\|_\tau \\
&\leq Ck^{-1/2}N^{1/2}\|G\|_\omega
\end{aligned}$$

where we have used the estimates of  $\omega^{-1}$ , standard inverse estimates and Hölder

inequalities. Similarly, we have

$$\|\omega^{1/2}E_\eta\|_{\Omega_s} \leq Ck^{-1/2}N^{1/2}\|G\|_\omega.$$

For  $\tau \in \Omega^N \setminus \Omega_s^N$ , we have

$$\begin{aligned} \|\omega^{1/2}E_\beta\|_\tau &\leq Ck^{-5/2}H_\tau^2[(\sigma_\beta^{-5/2} + \sigma_\beta^{-3/2}\sigma_\eta^{-1} + \sigma_\beta^{-1/2}\sigma_\eta^{-2})\max_\tau(\omega^{-1})_\beta^{1/2} + \sigma_\eta^{-3}\max_\tau\omega^{-1/2}]\|G\|_\tau \\ &+ Ck^{-2}\varepsilon^{-1/2}H_\tau^2[\sigma_\beta^{-2} + \sigma_\beta^{-1}\sigma_\eta^{-1} + \sigma_\eta^{-2}]\max_\tau\omega^{-1/2} \cdot \varepsilon^{1/2} \sum_{|\gamma|=1} \|D^\gamma G\|_\tau \\ &+ Ck^{-3/2}H_\tau[(\sigma_\beta^{-3/2} + \sigma_\beta^{-1/2}\sigma_\eta^{-1})\max_\tau(\omega^{-1})_\beta^{1/2} + \sigma_\eta^{-2}\max_\tau\omega^{-1/2}]\|G\|_\tau \\ &+ Ck^{-1}\varepsilon^{-1/2}H_\tau[\sigma_\beta^{-1} + \sigma_\eta^{-1}]\max_\tau\omega^{-1/2} \cdot \varepsilon^{1/2} \sum_{|\gamma|=1} \|D^\gamma G\|_\tau \\ &\leq Ck^{-1}\varepsilon^{-1/2}\ln^{-1}N\|G\|_\omega \end{aligned}$$

Similarly, we have

$$\|\omega^{1/2}E_\eta\|_{\Omega \setminus \Omega_s} \leq Ck^{-1}\varepsilon^{-1/2}\ln^{-1}N\|G\|_\omega.$$

□

**Lemma 4.** Let  $\tau \in \Omega_s^N$ . Let  $E = \omega^{-1}G - (\omega^{-1}G)^I$  where  $(\omega^{-1}G)^I$  denote the bilinear function that interpolates to  $\omega^{-1}G$  at the vertices of  $\tau$ . Then

$$\begin{aligned} \|\omega^{1/2}E\|_{\Omega_s} &\leq Ck^{-1}N^{-1/2}\|G\|_\omega \\ \|\omega^{1/2}E\|_{\Omega \setminus \Omega_s} &\leq Ck^{-1}\varepsilon^{1/2}\|G\|_\omega \end{aligned}$$

where  $H_\tau = \max\{h_{x,\tau}, h_{y,\tau}\}$ .

*Proof.* We make use of the following standard interpolation error bounds

$$\|u - u^I\|_{L^p(\tau)} \leq h_{x,\tau}^2\|u_{xx}\|_{L^p(\tau)} + h_{y,\tau}^2\|u_{yy}\|_{L^p(\tau)}$$

where  $p \in [1, \infty]$  and  $u \in C(\bar{\tau}) \cap W^{2,p}(\tau)$ .

Then, we have

$$\begin{aligned} \|E\|_\tau &\leq h_{x,\tau}^2\|(\omega^{-1}G)_{xx}\|_\tau + h_{y,\tau}^2\|(\omega^{-1}G)_{yy}\|_\tau \\ &\leq CH_\tau^2 \sum_{|\alpha|=2} \|D^\alpha(\omega^{-1}) \cdot G\|_\tau + CH_\tau^2\|(|(\omega^{-1})_\beta| + |(\omega^{-1})_\eta|) \cdot (|G_\beta| + |G_\eta|)\|_\tau \\ &\leq CH_\tau^2 \{ \|(\omega^{-1})_\beta G_\beta\|_\tau + \|(\omega^{-1})_\beta G_\eta\|_\tau + \|(\omega^{-1})_\eta G_\beta\|_\tau + \|(\omega^{-1})_\eta G_\eta\|_\tau \} \\ &+ CH_\tau^2 \sum_{|\alpha|=2} \|D^\alpha(\omega^{-1}) \cdot G\|_\tau. \end{aligned}$$

From the above inequality, we have

$$\|\omega^{1/2}E\|_{\Omega_s} \leq Ck^{-1}N^{-1/2}\|G\|_\omega$$



and

$$\|\omega^{1/2}E\|_{\Omega \setminus \Omega_s} \leq Ck^{-1}\varepsilon^{-1/2}N^{-1}\|G\|_{\omega}.$$

Following the techniques of (see[5, Lemma 4.4]), we have

$$(\omega^{1/2}E)(\mathbf{x}) = \int_{\mathbf{x}}^{\Gamma(\mathbf{x})} (\omega^{1/2}E)_{\eta} ds$$

where  $\mathbf{x} \in \Omega \setminus \Omega_s$ ,  $\Gamma(\mathbf{x}) \in \Gamma$  satisfies  $(\mathbf{x} - \Gamma(\mathbf{x})) \cdot \beta = 0$  and the following condition:

$$\begin{aligned} &\text{For } \forall \mathbf{y} \in \Gamma, (\mathbf{x} - \mathbf{y}) \cdot \beta = 0, \\ &|\mathbf{x} - \Gamma(\mathbf{x})| = \min_y |\mathbf{x} - \mathbf{y}|. \end{aligned}$$

From the above representation of  $\omega^{1/2}E$ , we have

$$\begin{aligned} \|\omega^{1/2}E\|_{\Omega_x}^2 &= \int_{\lambda_0}^{1-\lambda_y} \int_{l(\mathbf{x}_{lu}, \Gamma(\mathbf{x}_{lu}))} \left[ \int_{\mathbf{x}}^{\Gamma(\mathbf{x})} (\omega^{1/2}E)_{\eta} ds \right]^2 d\Omega \\ &+ \int_{1-\lambda_x}^1 \int_{l(\mathbf{x}_u, \Gamma(\mathbf{x}_u))} \left[ \int_{\mathbf{x}}^{\Gamma(\mathbf{x})} (\omega^{1/2}E)_{\eta} ds \right]^2 d\Omega + \int_0^{\lambda_0} \int_{l(\mathbf{x}_{ld}, \Gamma(\mathbf{x}_{ld}))} \left[ \int_{\mathbf{x}}^{\Gamma(\mathbf{x})} (\omega^{1/2}E)_{\eta} ds \right]^2 d\Omega \\ &\leq C\lambda_x^2 \left\{ \|(\omega^{1/2})_{\eta}E\|_{\Omega_x}^2 + \|\omega^{1/2}E_{\eta}\|_{\Omega_x}^2 \right\} \\ &\leq C\varepsilon^2 \ln^2 N \left\{ \sigma_{\eta}^{-2} \|\omega^{1/2}E\|_{\Omega_x}^2 + \|\omega^{1/2}E_{\eta}\|_{\Omega_x}^2 \right\} \\ &\leq Ck^{-2}\varepsilon^2 \ln^2 N \{ N^{3/2} \ln^{-2} N \cdot \varepsilon^{-1} N^{-2} + \varepsilon^{-1} \ln^{-2} N \} \|G\|_{\omega}^2 \\ &\leq Ck^{-2}\varepsilon \|G\|_{\omega}^2 \end{aligned}$$

where  $\lambda_0 = \frac{b_1}{b_2}\lambda_x$  and

- $x_{lu} \in \{(1 - \lambda_x, y) : \lambda_0 \leq y \leq 1 - \lambda_y\};$
- $x_{ld} \in \{(1 - \lambda_x, y) : 0 \leq y \leq \lambda_0\};$
- $x_u \in \{(x, 1 - \lambda_y) : 1 - \lambda_x \leq x \leq 1\}.$

□

**Lemma 5.** If  $\sigma_{\beta} = kN^{-1} \ln N$  and  $\sigma_{\eta} = k\tilde{\varepsilon}^{1/2} \ln N$ , where  $k > 1$  sufficiently large and independent of  $N$  and  $\varepsilon$ . Then

$$B((\omega^{-1}G)^I - \omega^{-1}G, G) \leq \frac{1}{16} \|G\|_{\omega}^2.$$

*Proof.* Cauchy-Schwarzs inequality gives

$$\begin{aligned} |B(E, G)| &\leq (\varepsilon + b^2\delta)^{1/2} \|\omega^{1/2}E_{\beta}\| \cdot (\varepsilon + b^2\delta)^{1/2} \|\omega^{-1/2}G_{\beta}\| + \hat{\varepsilon}^{1/2} \|\omega^{1/2}E_{\eta}\| \cdot \hat{\varepsilon}^{1/2} \|\omega^{-1/2}G_{\eta}\| \\ &+ C \|\omega^{1/2}E\| \cdot \|\omega^{-1/2}G_{\beta}\| + \|\omega^{1/2}E\| \cdot \|\omega^{-1/2}G\| \end{aligned}$$

From Lemma 3 and Lemma 4, we are done. □

**Theorem 3.1.** Assume that  $\sigma_\beta = kN^{-1} \ln N$  and  $\sigma_\eta = k\tilde{\varepsilon}^{1/2} \ln N$ , where  $k > 0$  is sufficiently large and independent of  $\varepsilon$  and  $N$ . Let  $\mathbf{x}^* \in \Omega \setminus \Omega_{xy}$ . Then for each nonnegative integer  $v$ , there exists a positive constant  $C = C(v)$  and  $K = K(v)$  such that

$$\|G\|_{W^{1,\infty}(\Omega_s \setminus \Omega'_0)} \leq CN^{-v},$$

$$\varepsilon|G|_{W^{1,\infty}((\Omega_x \cup \Omega_y) \setminus \Omega'_0)} + \|G\|_{L^\infty((\Omega_x \cup \Omega_y) \setminus \Omega'_0)} \leq CN^{-v}$$

and

$$\varepsilon|G|_{W^{1,\infty}(\Omega_{xy} \setminus \Omega'_0)} + \|G\|_{L^\infty(\Omega_{xy} \setminus \Omega'_0)} \leq C\varepsilon^{-1/2}N^{-v}.$$

*Proof.* On  $\Omega_s$ , we apply an inverse estimate.

On  $\Omega \setminus \Omega_s$  the application of an inverse estimate does not yield a satisfactory result, so we use a different technique.

Let  $\mathbf{x} \in \Omega_x \setminus \Omega'_0$  be arbitrary. Starting from  $\mathbf{x}$  we choose a polygonal curve  $\Gamma \subset (\Omega \setminus \Omega_{xy}) \setminus \Omega'_0$  that joints  $\mathbf{x}$  with some point on outflow boundaries. If  $(\mathbf{x} - \mathbf{x}^*) \cdot \boldsymbol{\eta} < 0$ , we can choose  $\Gamma$  as a line parallel to  $\beta$ . If  $(\mathbf{x} - \mathbf{x}^*) \cdot \boldsymbol{\eta} > 0$ , the situation is a little complicated. We can choose  $\Gamma$  as follows:

In  $\Omega_x \setminus \Omega'_0$ , we choose the direction of  $\Gamma$  along  $\boldsymbol{\eta}$  or the negative direction of  $x$ -axis so that  $\Gamma \cap \Omega_{xy} = \emptyset$ . In  $(\Omega_s \cup \Omega_y) \setminus \Omega'_0$ , we choose the direction of  $\Gamma$  along  $\boldsymbol{\eta}$  or the positive direction of  $y$ -axis.

Let  $T^N$  be the set of mesh rectangle  $\tau$  in  $(\Omega \setminus \Omega_{xy}) \setminus \Omega'_0$  that  $\Gamma$  intersects. Note that the length of the segment of  $\Gamma$  that lies in each  $\tau$  is at most  $C\varepsilon N^{-1} \ln N$  if  $\tau \in \Omega_x$  or  $\tau \in \Omega_y$ .

Then, by the fundamental theorem of calculus and inverse estimates in different domain, we can obtain the results.  $\square$

## References

- [1] T. Apel. *Anisotropic Finite Elements: Local Estimates and Applications*. B. G. Teubner Verlag, Stuttgart, 1999.
- [2] T. Apel and M. Dobrowolski. Anisotropic interpolation with applications to the finite element method. *Computing*, 47:277–293, 1992.
- [3] C. Johnson. *Numerical Solution of Partial Differential Equations by the Finite Element Method*. Cambridge University Press, Cambridge, 1987.
- [4] C. Johnson, A. H. Schatz, and L. B. Wahlbin. Crosswind smear and pointwise errors in streamline diffusion finite element methods. *Mathematics of Computation*, 49(179):25–38, 1987.
- [5] T. Linß and M. Stynes. The SDFEM on shishkin meshes for linear convection–diffusion problems. *Numer. Math.*, 87:457–484, 2001.